"Jumping of the nef cone for Fano varieties" (Totaro)

§ Introduction.

<u>Theorem</u> (Birkar - Cascini - Hacon - McKernan)

Let X be a $\mathbb{Q}$-factorial klt Fano variety.

Then $\mathrm{Cox}(X)$ is a f.g. $\mathbb{C}$-alg.

$$\mathrm{Cox}(X) = \bigoplus_{D \in \mathrm{Cl}(X)} H^0(X, \mathcal{O}(D))$$

Geometric meaning of $\mathrm{Cox}(X)$ being f.g.?

<u>Theorem</u> (Hu - Keel)

Let X be a normal projective variety.

Then $\mathrm{Cox}(X)$ is f.g.

$\Longleftrightarrow$ 1. X is $\mathbb{Q}$-factorial, $\mathrm{Cl}(X)$ is f.g.

2. $\mathrm{Nef}(X)$ is rational polyhedral, every nef divisor is semi-ample.

3. $\exists$ finitely many modifications $X \dashrightarrow X_i$ isom in codim 1. s.t. each X_i satisfies 1, 2,

$$\underline{\underline{\mathrm{Mov}(X)}} = \bigcup_{i=1}^{n} \mathrm{Nef}(X_i)$$

$\parallel$

the closure of the cone generated by line bdls L w/ $\mathrm{Bs}|L| \subset X$ codim ≥ 2

In this case, X is called a Mori dream space (MDS).

Remarks:

Since $X \dashrightarrow X_i$ is an isom in codim 1,

we have $\mathrm{Cl}(X) = \mathrm{Cl}(X_i)$,

$$H^0(X, \mathcal{O}(D)) = H^0(X_i, \mathcal{O}(D))$$

for all Weil divisors D on X,

$$\mathrm{Mov}(X) = \mathrm{Mov}(X_i),$$

$$\mathrm{Cox}(X) \simeq \mathrm{Cox}(X_i).$$

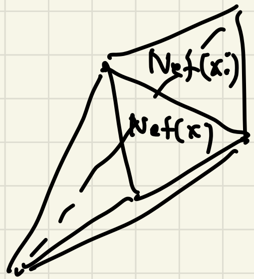
If $X \neq X_i$, then $\mathrm{Nef}(X) \neq \mathrm{Nef}(X_i)$

$$\mathrm{Nef}(X)^\circ \cap \mathrm{Nef}(X_i)^\circ = \phi.$$

Indeed, $L \in \mathrm{Nef}(X)^\circ \cap \mathrm{Nef}(X_i)^\circ$ is ample on X, X_i,

so $X = \operatorname{Proj} \bigoplus_m H^0(X, L^m)$

$$\cong \mathrm{Proj} \bigoplus H^0(X_i, L^n) = X_i$$



Mov(X)

Chamber decomposition of Mov(X).

Question:

Do the properties of Fano varieties as MDSs behave well under deformation?

Theorem (de Fernex - Hacon)

Let X_0 be a $\mathbb{Q}$-factorial terminal Fano variety.

Then $\mathrm{Cox}(X_0)$ deforms in a flat family:

if $X \to B$ is a flat family

w/ B smooth curve, fiber over $0 \in B \cong X_0$,

then $h^0(X, \mathcal{O}(D)) = h^0(X_t, \mathcal{O}(D))$ for all $D \in \mathrm{Cl}(X)$

for all $t \in B$ near 0.

Remark. $\mathrm{Cl}(X_0) = \mathrm{Cl}(X_t)$

because $\mathrm{Cl}(X_0) \xrightarrow{\sim} H_{2n-2}(X_0, \mathbb{Z})$ + Ehresmann's thm.

They also showed: $\overline{\mathrm{Eff}}(X_0) = \overline{\mathrm{Eff}}(X_t)$

$\mathrm{Mov}(X_0) = \mathrm{Mov}(X_t)$

Question (de Fernex - Hacon)

Let X_0 be a $\mathbb{Q}$-factorial terminal Fano variety. Does the chamber decomposition remain constant under any nearby flat deformation?

<u>Remark,</u> If true, then $\mathrm{Nef}(X_0)$ is constant.

Some positive results are known:

$\dim X_0 = 2$: $\mathrm{Mov}(X_0) = \mathrm{Nef}(X_0)$ ✓

<u>Theorem</u> (de Fernex - Hacon)

The question is true if either

$\dim X_0 \leq 3$

or $\dim X_0 \leq 4$ and X_0 is Gorenstein.

Theorem (Wiśniewski)

The nef cone of a smooth Fano variety remains constant under deformation.

Remark. The pf uses the hard Lefschetz thm.

Main Theorem (Totaro)

$\exists$ $\mathbb{Q}$-factorial terminal Fano 4-fold X_0

(resp. $\exists$ $\mathbb{Q}$-factorial terminal Gorenstein Fano 5-fold)

$\exists$ flat family $X \longrightarrow \mathbb{A}^1$ w/ fiber over $0 \cong X_0$

of Fano varieties

s.t. $\mathrm{Nef}(X_0) \subsetneq \mathrm{Nef}(X_t)$ for all $t \neq 0$

§ Construction

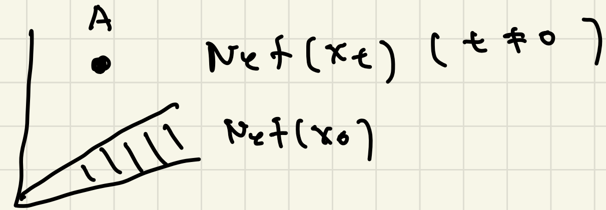
<u>Idea</u>: Construct a flip that deforms to an isomorphism:

$$\begin{tikzcd} X \arrow[rr, dashed, "\text{flip}"] \arrow[dr] & & Y \arrow[dl] \\ & Z \arrow[d] & \\ & \mathbb{A}^1 & \end{tikzcd} \qquad \begin{tikzcd} X_0 \arrow[rr, dashed, "\text{flip}"] \arrow[dr] & & Y_0 \arrow[dl] \\ & Z_0 & \end{tikzcd} \qquad X_t \xrightarrow{\sim} Y_t \quad (t \neq 0)$$

Take an ample divisor A on Y

Then $A|_{Y_0}$ is ample, so $A|_{X_0}$ is NOT ample.

$A|_{Y_t}$, $A|_{X_t}$ $(t \neq 0)$ are ample.



$\mathbb{Q}$-fact. terminal, Gorenstein Fano 5-fold

$a, b \in \mathbb{Z}_{>0}$. Assume $a > b > 1$.

We will construct a Fano $(a+b)$-fold with the required properties

($a=3$, $b=2$ gives the Fano 5-fold.)

local picture

$$(\mathbb{P}^a, \mathcal{O}(-1)^{b+1}) \dashrightarrow (\mathbb{P}^b, \mathcal{O}(-1)^{a+1})$$

$$([x_0,\dots,x_a], \xi_0,\dots,\xi_b) \longmapsto ([\xi_0,\dots,\xi_b], x_0,\dots,x_a)$$

$$\begin{tikzcd}
\cup & \arrow[d] & \arrow[d] & \supset \\
(0\text{-section}) & & Z & (0\text{-section}) \\
\mathbb{P}^a \arrow[u, phantom, "\shortparallel"] \arrow[rr] & & \ast \arrow[u, phantom, "\cup"] \arrow[d] & \mathbb{P}^b \arrow[u, phantom, "\shortparallel"] \arrow[l] \\
 & & \mathbb{A}^1 &
\end{tikzcd}$$

$V = \mathbb{C}^{b+1}, \quad W = \mathbb{C}^{a-b}$

$$Z = \{ f = (f_1, f_2) : V \longrightarrow V \oplus W,\ rk \leq 1 \}$$

There are two resolutions of Z:

$$X = \left\{ (f, L) : \begin{array}{l} L \subset V \oplus W \text{ a line,} \\ f : V \longrightarrow L \subset V \oplus W \end{array} \right\}$$

$$Y = \left\{ (f, S) \;:\; \begin{array}{l} S \subset V \ \text{codim } 1, \\ f : V/S \longrightarrow V \oplus W \end{array} \right\}$$

$$\left\{ \begin{array}{lll} \begin{array}{ccc} X^{a+b+1} & \longrightarrow & Z \\ \rotatebox{90}{$\in$} & & \rotatebox{90}{$\in$} \\ (f, h) & \longmapsto & f \end{array} & , & \begin{array}{l} \text{fiber over } (0 : V \longrightarrow V \oplus W) \\ \cong \mathbb{P}_*(V \oplus W) = \mathbb{P}^a \end{array} \\ & & \\ \begin{array}{ccc} Y & \longrightarrow & Z \\ \rotatebox{90}{$\in$} & & \rotatebox{90}{$\in$} \\ (f, S) & \longmapsto & f \end{array} & , & \begin{array}{l} \text{fiber over } (0 : V \longrightarrow V \oplus W) \\ = \mathbb{P}^*(V) = \mathbb{P}^b \end{array} \end{array} \right.$$

Define $t : Z \longrightarrow \mathbb{A}^1$, $f = (f_1, f_2) \longmapsto \operatorname{tr}(f_1)$

$$
\begin{tikzcd}
X \arrow[rr, dashed] \arrow[dr] & & Y \arrow[dl] \\
& Z \arrow[d, "t"] & \\
& \mathbb{A}^1 &
\end{tikzcd}
\qquad
\begin{tikzcd}
\mathbb{P}^a \subset X_0 \arrow[rr, dashed] \arrow[dr] \arrow[ddr] & & Y_0 \supset \mathbb{P}^b \arrow[dl] \arrow[ddl] \\
& Z_0 & \\
& \ni (0 : V \to V \oplus W) &
\end{tikzcd}
$$

Moreover

$$
\begin{array}{ccccc}
X & \longrightarrow & \mathbb{P}_*(V \oplus W) = \mathbb{P}^a, & \text{fiber over } L & \\
\rotatebox{90}{$\in$} & & \rotatebox{90}{$\in$} & = \mathrm{Hom}(V, L) & \\
(f, L) & \longmapsto & L & &
\end{array}
$$

$$\Rightarrow \quad X = \left(\underset{x_0, \dots, x_a}{\mathbb{P}^a}, \ \underset{\xi_0, \dots, \xi_b}{\mathcal{O}(-1)^{b+1}} \right) \quad \text{smooth}$$

$$ {}^t f = \begin{pmatrix} x_0 \xi_0 & \cdots & x_0 \xi_b \\ \vdots & & \vdots \\ x_a \xi_0 & \cdots & x_a \xi_b \end{pmatrix} $$

$$ {}^t (f, L) = \sum_{i=0}^{b} x_i \xi_i $$

$$ \Rightarrow \quad X_0 = \left\{ \sum_{i=0}^{b} x_i \xi_i = 0 \right\} \text{ in } X \quad : \text{ Gorenstein} $$

$$ \cup $$

$$ \mathrm{Sing}(X_0) = \{ x_0 = \cdots = x_b = \xi_0 = \cdots = \xi_b = 0 \} $$

$$ \cong \mathbb{P}^{a-b} $$

$X_0 \underset{\text{loc. around}}{=}$  $\times$ sm variety of dim $a-b-1$

a sing pt

sm quad, 2b. fold ≥ 4

: factorial

Similarly

$$\begin{array}{ccc} Y_0 & \longrightarrow & \mathbb{P}^*(V) = \mathbb{P}^b \\ \Downarrow & & \downarrow \\ (f, s) & \longmapsto & s \end{array}$$

, fiber over s

$= \{ f \in \mathrm{Hom}(V/s, V \oplus W) : \mathrm{tr}\, f_1 = 0 \}$

$= \mathrm{Hom}(V/s, S \oplus W)$

$\Rightarrow \quad Y_0 = (\mathbb{P}^b, T^* \oplus \mathcal{O}(-1)^{a-b})$

Global picture

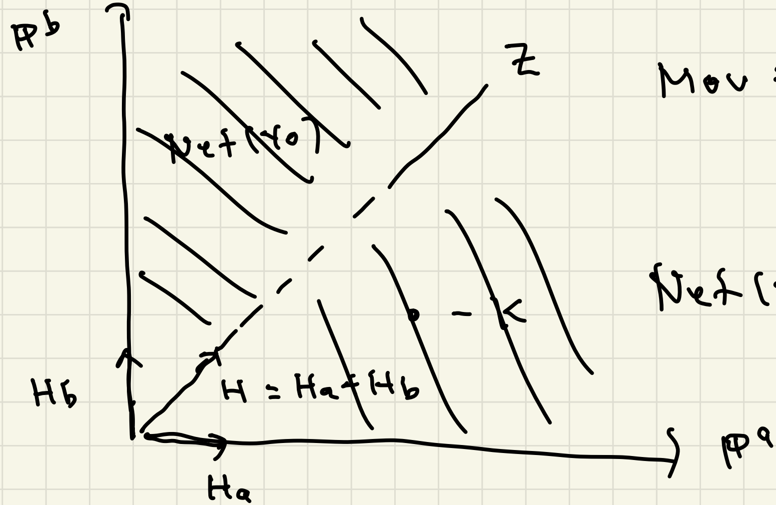
$X = \{ \sum_{i=0}^{b} x_i \xi_i = tg \}$ in $\mathbb{P}_{\mathbb{P}^a}(\mathcal{O}(1)^{b+1} \oplus \mathcal{O}) \times \mathbb{A}^1$

$x_0, \dots, x_a \quad \xi_0, \dots, \xi_b, g \qquad t$

$$\Rightarrow \begin{cases} X_0 = \{ \sum_{i=0}^{b} x_i \xi_i = 0 \} & \text{in } \mathbb{P}_{\mathbb{P}^a}(\mathcal{O}(1)^{b+1} \oplus \mathcal{O}) \\ X_t = \mathbb{P}_{\mathbb{P}^a}(\mathcal{O}(1)^{b+1}) \cong \mathbb{P}^a \times \mathbb{P}^b \\ (t \neq 0) \end{cases}$$

$$\text{Similarly } Y_0 = \mathbb{P}_{\mathbb{P}^b}(T \oplus \mathcal{O}(1)^{a-b} \oplus \mathcal{O}) \quad : \quad \begin{matrix} e(X_0) \\ \shortparallel \\ e(Y_0) = 2 \end{matrix}$$

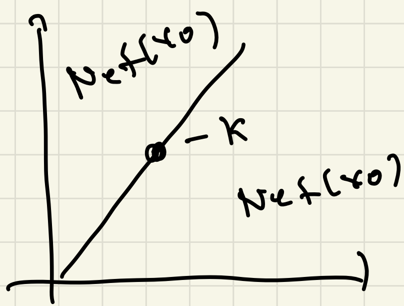
$$\therefore \begin{cases} -K_{X_0} = \underbrace{(a-b)}_{>0} H_a + (b+1) \underset{\text{tautological}}{\underline{\underline{H}}} & : \quad \text{Fano} \\ -K_{Y_0} = -(a-b) H_b + (a+1) H \end{cases}$$



$\mathrm{Mov} = \mathrm{Nef}(X_t) = \mathrm{Nef}(\mathbb{P}^a \times \mathbb{P}^b)$
$(t \neq 0)$

$\mathrm{Nef}(X_0)$

What happens if $a = b$?



X_0, Y_0 sm weak Fano.

$X_0 \dashrightarrow Y_0$ flop.

$\mathbb{Q}$-factorial terminal Fano 4-folds

$$(\mathbb{P}^2, \mathcal{O}(-1)^3) \overset{\text{flop}}{\dashrightarrow} (\mathbb{P}^2, \mathcal{O}(-1)^3) \quad \Big/ \quad \mathbb{Z}/2$$

$$= \quad X \overset{\text{flip}}{\dashrightarrow} Y$$